XIV. *On the Convergence of Infinite Series of Analytic Functions.*

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IN the first section of the following work an attempt is made to deal with the convergence of infinite series of functions defined by linear differential equations of the second order from the most general point of view. Functions of LAMÉ, BESSEL and LEGENDRE are considered as examples. In the second section the results obtained are applied to the expansion of an arbitrary uniform analytic function of z in a series of hypergeometric functions, and the expansion is shown to be valid if the function is regular within a certain ellipse in the z-plane. An expansion in a series of LEGENDRE'S associated functions is deduced by a transformation. The method has been applied by the writer to other cases, but the foregoing offer adequate illustration of the general theory.

SECTION I.—GENERAL THEOREMS.

§ 1. *Theorem I.*—Consider the differential equation

$$\frac{d^2y}{dz^2}+k^2Qy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where

$$Q = Q_0+\frac{Q_1}{k}+\frac{Q_2}{k^2}+\ldots,$$

k is a large constant, $\surd Q_0$, Q_1, $Q_2\ldots$ are analytic functions of z, independent of k, without singularities or branch-points so long as z lies within a given simply-connected region S in the z-plane, though $\surd Q_0$, Q_1, $Q_2\ldots$ may have singularities on the boundary of S; and the series defining Q converges if $|k|>\mathrm{R}$, so long as z is in the region S.

Let $z = a$ be a point within S.

Consider the particular integral of (1) defined, when $z = a$, by

$$y = \alpha_0 + \frac{\alpha_1}{k} + \frac{\alpha_2}{k^2} + \ldots,$$

$$\frac{dy}{dz} = k\left(\beta_0 + \frac{\beta_1}{k} + \frac{\beta_2}{k^2} + \ldots\right),$$

where α and β are constants, and both series converge if $|k| > \mathrm{R}$.

The expression

$$y = e^{ik\omega}\left(\phi_0 + \frac{\phi_1}{k} + \frac{\phi_2}{k^2} + \ldots ad\ inf.\right)$$

$$+ e^{-ik\omega}\left(\psi_0 + \frac{\psi_1}{k} + \frac{\psi_2}{k^2} + \ldots ad\ inf.\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where ω, ϕ, ψ are functions of z independent of k, which can be constructed as a formal representation of this integral, is for values of k such that $|k| > \mathrm{R}$, and for values of z within the region S, a convergent series and consequently a true representation of the integral considered.

Further, when k is very large,

$$e^{ik\omega}\phi_0 + e^{-ik\omega}\psi_0$$

is an approximate value of the integral, whether the integral and the approximate value increase indefinitely with k or not.

This proposition has been proved by HORN* for functions of a real variable z; the proof for functions of a complex variable is similar; a brief outline is as follows:—

Let $v_1 = e^{ik\omega}\phi_0$, $v_2 = e^{-ik\omega}\psi_0$ be two independent integrals of the equation

$$\frac{d^2y}{dz^2} + k^2\mathrm{Q}_0 y = 0.$$

Write

$$\mathrm{D}(u) \equiv \frac{\begin{vmatrix} u & u' & u'' \\ v_1 & v'_1 & v''_1 \\ v_2 & v'_2 & v''_2 \end{vmatrix}}{\begin{vmatrix} v_1 & v'_1 \\ v_2 & v'_2 \end{vmatrix}}.$$

Then

$$u'' + k^2\mathrm{Q}u \equiv \mathrm{D}(u) + \mathrm{E}(u),$$

where

$$\mathrm{E}(u) = \frac{\mathrm{S}}{k}u' + \mathrm{T}u,$$

S and T being functions of z and k, developable in convergent series of powers of k^{-1}, if $|k| > \mathrm{R}$.

* 'Mathematische Annalen,' vol. 52, p. 345 (1899).

If, then, we integrate the equations

$$\mathrm{D}(u_0) = 0,$$
$$\mathrm{D}(u_m)+\mathrm{E}(u_{m-1}) = 0 \quad (m = 1, 2\ldots),$$

with the initial conditions

$$\bar{u}_0 = \alpha_0+\frac{\alpha_1}{k}+\ldots, \quad \bar{u}'_0 = k\left(\beta_0+\frac{\beta_1}{k}+\ldots\right),$$

$$\bar{u}_m = 0, \quad \bar{u}'_m = 0 \quad (m = 1, 2\ldots),$$

we shall have

$$y = u_0+u_1+u_2+\ldots,$$

provided this series converges.

We find that

$$u_m = -\frac{e^{ik\omega}\phi_0}{2ik}\int_a^z \frac{e^{-ik\omega}\psi_0\mathrm{E}(u_{m-1})\,dz}{\omega'\phi_0\psi_0-\dfrac{\phi_0\psi'_0-\phi'_0\psi_0}{2ik}}$$
$$+\frac{e^{-ik\omega}\psi_0}{2ik}\int_a^z \frac{e^{ik\omega}\phi_0\mathrm{E}(u_{m-1})\,dz}{\omega'\phi_0\psi_0-\dfrac{\phi_0\psi'_0-\phi'_0\psi_0}{2ik}},$$

the integrals being taken along any finite path within the region S.

It can be proved by induction that either $|e^{ik\omega}u_m|$ or $|e^{-ik\omega}u_m|$

$$\ngtr \frac{\mathrm{M}^{2m+1}}{(2r)^m}\cdot\frac{|z-a|^m}{m!}\cdot(1+3^m),$$

according as $|e^{-ik\omega}|$ is greater or less than unity; where M is a finite real positive quantity, independent of k.

Hence the series

$$y = u_0+u_1+u_2+\ldots,$$

when multiplied by either $e^{ik\omega}$ or $e^{-ik\omega}$, according to the value of $\arg(k\omega)$, is absolutely and uniformly convergent for all values of k such that $|k| > \mathrm{R}$.

§ 2. A slight change of notation is convenient. Consider the equation

$$\frac{d^2y}{dz^2}+\mu_n(z)\,y = 0 \quad \ldots\ldots\ldots\ldots \quad (3),$$

where n is a positive integer, $\mu_n(z)$ is expansible in the series

$$n^r\left\{f_0(z)+\frac{1}{n^a}f_1(z)+\ldots\right\},$$

arranged in descending powers of n and convergent if z is confined to a finite simply-

connected area C in the z-plane, within which $\mu_n(z)$ is regular, though $\mu_n(z)$ may have singularities on the boundary of C.

r is a positive quantity, and $\sqrt{f_0(z)}$ as well as $f_0(z)$ must be regular within C.

Then the approximate value of any solution of the equation (3) for large values of n is

$$y = A_1\psi_1(z) \, . \, e^{in^{\frac{1}{2}r}\int\sqrt{f_0(z)}\,.\,dz} + A_2\psi_2(z) \, . \, e^{-in^{\frac{1}{2}r}\int\sqrt{f_0(z)}\,.\,dz} \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where A_1 and A_2 are arbitrary constants, and $\psi_1(z)$, $\psi_2(z)$, are functions of z that remain finite as n increases indefinitely.

This follows from Theorem I.

Unless $\mu_n(z)$ possesses a line of singularities everywhere dense, forming a closed curve, the result holds for all values of z except the singularities and branch-points of $\mu_n(z)$.

Denote the solutions whose approximate values are

$$\psi_1(z)\, e^{in^{\frac{1}{2}r}\int\sqrt{f_0(z)}\,.\,dz} \quad \text{and} \quad \psi_2(z) \, . \, e^{-in^{\frac{1}{2}r}\int\sqrt{f_0(z)}\,.\,dz}$$

by

$$p_n(z) \quad \text{and} \quad q_n(z).$$

Denote

$$e^{i\int\sqrt{f_0(z)}\,.\,dz} \quad \text{by} \quad \theta(z).$$

Theorem II.—The series

$$F(x, t) \equiv \sum_{n=1}^{\infty} c_n p_n(x) \, . \, q_n(t) \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

the c's being arbitrary save for the condition that the series $\sum_{n=1}^{\infty} c_n z^n$ has unit radius of convergence, is or is not absolutely convergent according as $|\theta(x)|$ is less than or is greater than $|\theta(t)|$.

If $|\theta(x)| = |\theta(t)|$, $F(x, t)$ converges or diverges according as $\sum_{n=1}^{\infty} c_n$ converges or diverges.

For the n^{th} term of the series (5) is approximately equal to

$$c_n \, . \, \psi_1(x) \, . \, \psi_2(t) \, . \left\{\frac{\theta(x)}{\theta(t)}\right\}^{n^{\frac{1}{2}r}},$$

of which the modulus is

$$|\psi_1(x) \, . \, \psi_2(t)| \times |c_n| \, . \left\{\left|\frac{\theta(x)}{\theta(t)}\right|^{\frac{1}{2}r}\right\}^n;$$

whence the result.

The examination of some special equations will illustrate Theorems I. and II.

§ 3. For LAMÉ's equation

$$\frac{1}{y}\frac{d^2y}{dz^2} = n(n+1)k^2\text{sn}^2z + B \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\omega = \pm ik\int \text{sn}z \, . \, dz = \mp i \log(\text{dn}z + k\text{cn}z),$$

and the general solution of LAMÉ's equation for large values of n is approximately

$$\mathrm{sn}^{-\frac{1}{2}}z[\mathrm{C}_1(\mathrm{dn}z+k\mathrm{cn}z)^{n+\frac{1}{2}}+\mathrm{C}_2(\mathrm{dn}z+k\mathrm{cn}z)^{-n-\frac{1}{2}}] \quad \ldots \ldots \quad (7),$$

where C_1 and C_2 are arbitrary constants.

In the notation of BYERLY*

$$\mathrm{E}_n{}^p(\mathrm{sn}z) = \mathrm{C}_1\mathrm{sn}^{-\frac{1}{2}}z(\mathrm{dn}z+k\mathrm{cn}z)^{n+\frac{1}{2}}$$

and

$$\mathrm{F}_n{}^p(\mathrm{sn}z) = \mathrm{C}_2\mathrm{sn}^{-\frac{1}{2}}z(\mathrm{dn}z+k\mathrm{cn}z)^{-n-\frac{1}{2}}$$

approximately, for large values of n, where

$$\mathrm{B} \equiv p(1+k^2).$$

Hence the series

$$\sum_{n=1}^{\infty} c_n \mathrm{E}_n{}^p(\mathrm{sn}x) \,.\, \mathrm{F}_n{}^p(\mathrm{sn}t)$$

corresponding to LAMÉ's equation converges if

$$|\mathrm{dn}x+k\mathrm{cn}x| < |\mathrm{dn}t+k\mathrm{cn}t|.$$

Write

$$x = \xi+i\eta, \quad t = \xi_1+i\eta_1.$$

The condition reduces to

$$\frac{\mathrm{dn}(\xi, k)\,.\,\mathrm{dn}(\eta, k')}{\mathrm{cn}(\xi, k)} < \frac{\mathrm{dn}(\xi_1, k)\,.\,\mathrm{dn}(\eta_1, k')}{\mathrm{cn}(\xi_1, k)} \quad \ldots \ldots \quad (8).$$

§ 4. The equation of the elliptic cylinder

$$\frac{1}{y}\frac{d^2y}{d\phi^2} = n(n+1)\cos^2\phi+\mathrm{B} \quad \ldots \ldots \quad (9)$$

is, in the notation of HEINE,† satisfied by the functions

$$\mathrm{E}_n(\phi) \quad \text{and} \quad \mathrm{F}_n(\phi).$$

When n is large,

$$\mathrm{E}_n(\phi) = \mathrm{C}_1\sqrt{\sec\phi}\,.\,e^{(n+\frac{1}{2})\sin\phi}$$

and

$$\mathrm{F}_n(\phi) = \mathrm{C}_2\sqrt{\sec\phi}\,.\,e^{-(n+\frac{1}{2})\sin\phi}, \text{ approximately.}$$

Hence the series

$$\sum_{n=1}^{\infty} c_n \mathrm{E}_n(x) \,.\, \mathrm{F}_n(t)$$

corresponding to the equation of the elliptic cylinder converges if

$$|e^{\sin x-\sin t}| < 1.$$

* 'FOURIER'S Series and Spherical Harmonics,' p. 255.

† 'Kugelfunctionen,' 2nd edition, p. 404.

Let

$$x = \xi + i\eta, \quad t = \xi_1 + i\eta_1.$$

The condition is that

$$\sin\xi \cosh\eta < \sin\xi_1 \cosh\eta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

§ 5. The equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left\{(n+\alpha)(n+\alpha+1) - \frac{(n+\beta)^2}{1-z^2}\right\} y = 0 \quad . \quad . \quad . \quad (11)$$

is satisfied by LEGENDRE's associated functions

$$Q_{n+\alpha}{}^{-n-\beta}(z) \quad \text{and} \quad Q_{-n-\alpha-1}{}^{n+\beta}(z).$$

We find that when n is a large positive integer

$$Q_{n+\alpha}{}^{-n-\beta}(z) = e^{-(n+\beta)\frac{1}{2}\pi i}\frac{\Pi(\alpha-\beta)\,.\,\pi^{\frac{1}{2}}}{\Pi(n+\alpha+\frac{1}{2})} 2^{-n-\alpha-1} z^{-\alpha+\beta-1}(1-z^2)^{-\frac{1}{2}n-\frac{1}{2}\beta}$$

and

$$Q_{-n-\alpha-1}{}^{n+\beta}(z) = e^{(n+\beta)\frac{1}{2}\pi i}\frac{\Pi(\beta-\alpha-1)\,.\,\pi^{\frac{1}{2}}}{\Pi(-n-\alpha-\frac{1}{2})} 2^{n+\alpha} z^{\alpha-\beta}(1-z^2)^{\frac{1}{2}n+\frac{1}{2}\beta}$$

approximately.

Whence

$$Q_{-n-\alpha-1}{}^{n+\beta}(x)\, Q_{n+\alpha}{}^{-n-\beta}(t) = \frac{(-1)^n}{2n+2\alpha+1}\cdot\frac{\pi\sin(\alpha+\frac{1}{2})\pi}{\sin(\alpha-\beta)\pi}\frac{1}{t}\left(\frac{x}{t}\right)^{\alpha-\beta}\left(\frac{1-x^2}{1-t^2}\right)^{\frac{1}{2}n+\frac{1}{2}\beta}.$$

Accordingly the series

$$\sum_{n=1}^{\infty} c_n Q_{-n-\alpha-1}{}^{n+\beta}(x)\,.\,Q_{n+\alpha}{}^{-n-\beta}(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

converges if

$$|1-x^2| < |1-t^2|$$

and diverges if

$$|1-x^2| > |1-t^2|;$$

the c's being arbitrary except that the power series $\sum_{n=1}^{\infty} c_n z^n$ has unit radius of convergence.

If

$$|1-x^2| = |1-t^2|,$$

the convergence depends on the values of the c's; we shall exclude this case.

Let

$$x = \xi + i\eta, \quad t = \xi_1 + i\eta_1.$$

$$|1-x^2| < |1-t^2|,$$

if

$$(\xi^2+\eta^2)^2 - 2(\xi^2-\eta^2) < (\xi_1{}^2+\eta_1{}^2)^2 - 2(\xi_1{}^2-\eta_1{}^2)$$

Now let us choose the c's according to the following law. Unless n is an integral power of 3, $c_n = 0$; if

$$n = 3^m, \quad c_n = \frac{(-1)^m}{2m+1}.$$

For the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{3^n},$$

the circle of convergence is a cut, and the function defined by the series is regular within the circle, but has a line of singularities round the circle that are everywhere dense.

Hence the series

$$\sum_{n=1}^{\infty} \mathrm{Q}_{3^n}{}^{-3^n-\frac{1}{2}}(1\tfrac{1}{4}) \,.\, \mathrm{Q}_{-3^n-1}{}^{3^n+\frac{1}{2}}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

defines a function that is regular within the two loops of the oval of CASSINI

$$(\xi^2+\eta^2)^2-2(\xi^2-\eta^2)+\tfrac{175}{256} = 0,$$

where $x = \xi+i\eta$, which has singularities everywhere dense round the two loops, and which consequently cannot be continued across the boundary of either of the loops.

Hence we have constructed an analytic function that exists within two separate regions, is the same function in both regions, but cannot be analytically continued from one region into the other.

§ 6. Consider the equation

$$\frac{d^2y}{dz^2} = p^2n^2yz^{2p-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

which is satisfied by the Bessel functions

$$z^{\frac{1}{2}}\mathrm{J}_{\pm(2p)^{-1}}(inz^p).$$

n is a positive integer, p is real, and the cases $p = 0$ and $(2p)^{-1}$ an integer are excluded.

Take as the standard solutions

$$p_n(z) = iz^{\frac{1}{2}}\{e^{-\frac{i\pi}{4p}}.\mathrm{J}_{-\frac{1}{2p}}(inz^p)-e^{\frac{i\pi}{4p}}.\mathrm{J}_{\frac{1}{2p}}(inz^p)\}$$

and

$$q_n(z) = z^{\frac{1}{2}}\{e^{\frac{i\pi}{4p}}.\mathrm{J}_{-\frac{1}{2p}}(inz^p)-e^{-\frac{i\pi}{4p}}.\mathrm{J}_{\frac{1}{2p}}(inz^p)\}.$$

From the asymptotic expansions of the Bessel functions, or from Theorem I., we find that when n is large

$$p_n(z) = \sqrt{\frac{2}{n\pi}}\,.\,\sin\frac{\pi}{2p}\,.\,z^{\frac{1}{2}-\frac{1}{2}p}\,.\,e^{nz^p}$$

and

$$q_n(z) = \sqrt{\frac{2}{n\pi}}\,.\,\sin\frac{\pi}{2p}\,.\,z^{\frac{1}{2}-\frac{1}{2}p}\,.\,e^{-nz^p}$$

approximately.

Hence the series of Bessel functions

$$\sum_{n=1}^{\infty} c_n p_n(x) q_n(t),$$

where the c's are arbitrary, save that the series $\sum_{n=1}^{\infty} c_n z^n$ has unit radius of convergence, converges if

$$|e^{x^p - t^p}| < 1.$$

To interpret this condition geometrically, let

$$x = re^{i\theta}, \quad t = \rho e^{i\alpha},$$

and the condition reduces to

$$r^p \cos p\theta < \rho^p \cos p\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

§ 7. *If we are given any finite simply-connected plane area whose boundary either is an analytic curve or is made up of portions of a finite number of analytic curves, the interior can be conformally represented by the interior of a circle, any given point corresponding to the centre of the circle, and the equation of the boundary can be expressed in Cartesian co-ordinates (ξ, η) in the form $|\theta(z)| = \text{constant}$, where $z \equiv \xi + i\eta$, and $\theta(z)$ is a uniform analytic function of z.

The following is a converse of Theorem II.

Theorem III.—We can construct a series, $\mathrm{F}(x, t)$, which converges if x lie within the area bounded by $|\theta(z)| = |\theta(t)|$, but not if x lie outside the area.

Points lying on the boundary of the area are excluded.

$$\mathrm{F}(x, t) = \sum_{n=1}^{\infty} c_n f_n(x) g_n(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

where $c_1, c_2 \ldots c_n \ldots$ are arbitrary, save for the restriction that $\sum_{n=1}^{\infty} c_n z^n$ has unit radius of convergence, and $f_n(z)$, $g_n(z)$ are solutions of the linear differential equation of the second order

$$\frac{d^2y}{dz^2} + n^r y \left\{ -\left[\frac{1}{\theta(z)} \frac{d\theta(z)}{dz}\right]^2 + \frac{1}{n^\alpha} f_1(z) + \ldots \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where r, α are real positive constants, independent of n, $f_1(z), f_2(z) \ldots$ are functions of z regular within the curve $|\theta(z)| = |\theta(t)|$, and the series in the bracket is arranged in descending powers of n and is convergent when n is large.

§ 8. *Theorem IV.*—Let $\phi(z)$ be a solution of the linear differential equation of the n^{th} order

$$(a_0 + a_1 z + \ldots + a_n z^n) \frac{d^n y}{dz^n} + (b_0 + b_1 z + \ldots + b_{n-1} z^{n-1}) \frac{d^{n-1} y}{dz^{n-1}}$$

$$+ \ldots + (k_0 + k_1 z) \frac{dy}{dz} + l_0 y = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

the coefficient of $d^r y/dz^r$ being a polynomial in z of order r.

* Forsyth, 'Theory of Functions,' 2nd edition, chapter XX.

Then

$$f(z) = \int_c \frac{\phi(t)}{t-z} dt,$$

taken along a suitable path, satisfies the equation (18), and there are in general $n-1$ such integrals, linearly independent.

For

$$\frac{d^n f(z)}{dz^n} = n! \int_c \frac{\phi(t)}{(t-z)^{n+1}} dt$$

$$= \int_c \frac{\phi^{(n)}(t)}{t-z} dt - \left[\frac{\phi^{(n-1)}(t)}{t-z} + \frac{\phi^{(n-2)}(t)}{(t-z)^2} + \ldots + (n-1)! \frac{\phi(t)}{(t-z)^n}\right]$$

on integrating by parts, where $\phi^{(n)}(t)$ denotes $d^n\phi(t)/dt^n$.

Let

$$\psi_n(z, t) \equiv \frac{\phi^{(n-1)}(t)}{t-z} + \frac{\phi^{(n-2)}(t)}{(t-z)^2} + \ldots + (n-1)! \frac{\phi(t)}{(t-z)^n}.$$

Substitute in the left side of (18), and subtract the zero quantity

$$\int_c \frac{1}{t-z} \cdot \{(a_0 + a_1 t + \ldots + a_n t^n)\, \phi^{(n)}(t) + \ldots + l_0 \phi(t)\} \,.\, dt,$$

and the expression becomes the integral of a perfect differential.

Hence

$$f(z) = \int_c \frac{\phi(t)}{t-z} dt$$

satisfies the equation (18), provided that

$$\begin{aligned}
&(a_0 + a_1 z + \ldots + a_n z^n)\, \psi_n(z, t) + \ldots + (k_0 + k_1 z)\, \psi_1(z, t) \\
&\quad + \{(a_1 + \ldots + a_n z^{n-1})\, \phi^{(n-1)}(t) + (a_2 + a_3 z + \ldots + a_n z^{n-2})\left[t\phi^{(n-1)}(t) - \phi^{(n-2)}(t)\right] \\
&\quad + \ldots + a_n\left[t^{n-1}\phi^{(n-1)}(t) - (n-1)\, t^{n-2}\phi^{(n-2)}(t) + \ldots + (-1)^{n-1} . (n-1)!\, \phi(t)\right]\} \\
&\quad + \ldots \\
&\quad + k_1 \phi(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)
\end{aligned}$$

has the same value at the beginning and the end of the path.

Let the roots of the equation

$$a_0 + a_1 z + \ldots + a_n z^n = 0,$$

supposed all different, be denoted by

$$\lambda_1, \quad \lambda_2, \ldots \lambda_n.$$

The points

$$z = \lambda_1, \quad z = \lambda_2, \ldots z = \lambda_n, \quad \text{and} \quad z = \infty$$

are the only possible singularities or branch-points of $\phi(z)$ and its derivates, and sc

$$t = \lambda_1 \dots t = \lambda_n, \quad t = \infty \quad \text{and} \quad t = z,$$

are the only possible singularities or branch-points of the expression (19).

Hence, if we take a "Doppelumlauf" round two of the points

$$t = \lambda_1, \quad t = \lambda_2, \dots t = \lambda_n,$$

we have a contour satisfying the condition. There are in general $n-1$ independent contours of this kind, which, together with $\phi(z)$, which is given by a simple contour round $t = z$, form a complete set of integrals of the equation (18).

Some of the contours may become evanescent, either because the equation

$$a_0 + a_1 z + \dots + a_n z^n = 0$$

has equal roots, or because

$$z = \lambda_1, \quad z = \lambda_2, \dots z = \lambda_n$$

are not all singularities or branch-points of $\phi(z)$. In that case the method does not yield a complete set of integrals. In special cases a contour may reduce to a straight path connecting two singularities.

§ 9. The theorem has been proved for the special case of the hypergeometric equation in a posthumous paper of JACOBI.*

The result is of importance in the expansion of $(t-x)^{-1}$ in a series of hypergeometric functions.

JACOBI proves further that if n is a positive integer

$$\mathrm{F}(-n, p+n, \gamma, x) = \frac{\Pi(\gamma-1)}{\Pi(n+\gamma-1)} x^{1-\gamma}(1-x)^{\gamma-p} \frac{d^n}{dx^n}\{x^{n+\gamma-1}(1-x)^{n+p-\gamma}\} \quad . \quad (20),$$

from which it follows after integration by parts that if the real parts of $\gamma+n$ and $p-\gamma+n+1$ are positive,

$$\int_0^1 \frac{z^{\gamma-1}(1-z)^{p-\gamma}}{z-t} \mathrm{F}(-n, p+n, \gamma, z)\, dz$$
$$= (-t)^{-n-1} \frac{\Pi(n)\Pi(\gamma-1)\Pi(p-\gamma+n)}{\Pi(p+2n)} \mathrm{F}\left(n+1, \gamma+n, p+2n+1, \frac{1}{t}\right). \quad . \quad (21).$$

§ 10. *Theorem V.*—Consider the equations

$$\theta_1(z)\frac{d^2u}{dz^2} + \theta_2(z)\frac{du}{dz} + \alpha\theta_3(z)u = 0,$$

and

$$\theta_1(z)\frac{d^2v}{dz^2} + \theta_2(z)\frac{dv}{dz} + \beta\theta_3(z)v = 0,$$

* "Untersuchungen über die Differentialgleichungen der hypergeometrischen Reihe," 'Crelle,' vol. 56. See also JACOBI'S 'Gesammelte Werke,' vol. VI., pp. 184–202.

θ_1, θ_2, θ_3 being any uniform analytic functions of z without singularities in the finite part of the z-plane, and α, β two unequal constants. The integral

$$\int_{\mathrm{L}} uv \frac{\theta_3(z)}{\theta_1(z)} e^{\int \frac{\theta_2(z)}{\theta_1(z)} dz} dz$$

vanishes if L is a suitably chosen path.

For the integral

$$= \frac{1}{\alpha-\beta}\left[e^{\int \frac{\theta_2(z)}{\theta_1(z)} dz}\left(u\frac{dv}{dz}-v\frac{du}{dz}\right)\right],$$

and vanishes if the expression in square brackets has the same value at the beginning and the end of the path. A suitable path is, therefore, a "Doppelumlauf" round two of the roots of $\theta_1(z) = 0$ if these roots are not all equal; the path may in special cases reduce to a straight line.

§ 11. The special case of the hypergeometric function was considered by JACOBI, who proved in the paper already quoted that if the real parts of γ and $p+1-\gamma$ are positive,

$$\int_0^1 \mathrm{F}(-m, p+m, \gamma, z)\,\mathrm{F}(-n, p+n, \gamma, z)\,z^{\gamma-1}(1-z)^{p-\gamma}\,dz = 0 \text{ if } m \neq n \quad . \quad (22).$$

JACOBI proved further that, under the same conditions,

$$\int_0^1 [\mathrm{F}\{-n, p+n, \gamma, z\}]^2 z^{\gamma-1}(1-z)^{p-\gamma}\,dz = \frac{1}{p+2n}\cdot\frac{\Pi(n)\{\Pi(\gamma-1)\}^2\,\Pi(p-\gamma+n)}{\Pi(p+n-1)\,\Pi(\gamma+n-1)} \quad (23).$$

An important special case of (22) is given by $m = 0$, in which case, under the same conditions,

$$\int_0^1 \mathrm{F}(-n, p+n, \gamma, z)\,z^{\gamma-1}(1-z)^{p-\gamma}\,dz = 0, \text{ unless } n = 0 \quad . \quad . \quad . \quad (24).$$

We are now in a position to investigate the expansion of an arbitrary uniform analytic function in a series of hypergeometric functions.

SECTION II.—HYPERGEOMETRIC FUNCTIONS.

§ 12. The hypergeometric function

$$\mathrm{F}(\alpha, \beta, \gamma, x)$$

is an analytic function of x for all values of x, with branch-points at

$$x = 0,\ 1, \text{ and } \infty.$$

If $|x| < 1$, one of the branches of the function can be represented by the series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha.(\alpha+1).\beta.(\beta+1)}{1.2.\gamma.(\gamma+1)}x^2 + \ldots ad\ inf.$$

§ 13. The formula

$$\alpha(\gamma-\beta)(\beta-\alpha+1)\,\mathrm{F}(\alpha+1,\ \beta-1,\ \gamma,\ x)-\beta(\gamma-\alpha)(\alpha-\beta+1)\,\mathrm{F}(\alpha-1,\ \beta+1,\ \gamma,\ x)$$
$$=(\alpha-\beta)\{x(\overline{\alpha-\beta}^2-1)+2\alpha\beta+\gamma(1-\alpha-\beta)\}\,\mathrm{F}(\alpha,\ \beta,\ \gamma,\ x) \quad . \quad . \quad . \quad (25)$$

may be easily verified.*

Let α be a negative integer; $\mathrm{F}(\alpha,\ \beta,\ \gamma,\ x)$ becomes a polynomial in x. Write $\alpha=-n$, $\alpha+\beta=p$. We deduce from (25) the result†

$$\Pi(p-1)\,\Pi(\gamma-p-1)\,\Pi(\gamma-1)\,p\,(x-z)\,\mathrm{F}(0,\ p,\ \gamma,\ z)\,\mathrm{F}(0,\ p,\ \gamma,\ x)$$
$$-\frac{1}{\Pi(1)}\,\Pi(p)\,\Pi(\gamma-p-2)\,\Pi(\gamma)\,(p+2)\,(x-z)\,\mathrm{F}(-1,\ p+1,\ \gamma,\ z)\,\mathrm{F}(-1,\ p+1,\ \gamma,\ x)$$
$$+\ldots+\frac{(-1)^n}{\Pi(n)}\,.\,\Pi(p+n-1)\,\Pi(\gamma-p-n-1)\,\Pi(\gamma+n-1)\,(p+2n)\,(x-z)$$
$$\mathrm{F}(-n,\ p+n,\ \gamma,\ z)\,\mathrm{F}(-n,\ p+n,\ \gamma,\ x)$$

is equal to

$$\frac{(-1)^{n+1}}{\Pi(n)}\,\frac{\Pi(p+n)\,.\,\Pi(\gamma-p-n-1)\,.\,\Pi(\gamma+n)}{p+2n+1}\,[\mathrm{F}(-n,\ p+n,\ \gamma,\ z)\,\mathrm{F}(-n-1,\ p+n+1,\ \gamma,\ x)$$
$$-\mathrm{F}(-n,\ p+n,\ \gamma,\ x)\,\mathrm{F}(-n-1,\ p+n+1,\ \gamma,\ z)] \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Assume that the real parts of γ and $(p+1-\gamma)$ are positive. Multiply (26) through by

$$\frac{z^{\gamma-1}(1-z)^{p-\gamma}}{(z-x)(z-t)}\,dz$$

and integrate from $z=0$ to $z=1$. Making use of (22) and (24), we find after some reduction that, when the real parts of γ and $(p+1-\gamma)$ are positive,

$$\frac{1}{t}\,\mathrm{F}(0,\ p,\ \gamma,\ x)\,.\,\mathrm{F}\left(1,\ \gamma,\ p+1,\ \frac{1}{t}\right)$$
$$-\frac{\gamma}{p+1}\,\frac{1}{t^2}\,\mathrm{F}(-1,\ p+1,\ \gamma,\ x)\,\mathrm{F}\left(2,\ \gamma+1,\ p+3,\ \frac{1}{t}\right)+\ldots$$
$$+(-1)^n\,\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{(p+n)(p+n+1)\ldots(p+2n-1)}\,\frac{1}{t^{n+1}}\,\mathrm{F}(-n,\ p+n,\ \gamma,\ x)$$
$$\mathrm{F}\left(n+1,\ \gamma+n,\ p+2n+1,\ \frac{1}{t}\right)$$

* It may be derived at once from formulæ [1], [2], [3], [6], and [7], given by GAUSS, "Disquisitiones generales circa seriem infinitam

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}\,.\,x+\ldots,"$$

'Werke,' vol. III., pp. 125–162.

Reference will be made also to another memoir, "Determinatio serici nostræ per æquationem differentialem secundi ordinis," GAUSS' 'Werke,' vol. III., pp. 207–230.

† For the method of deduction, *cf.* WHITTAKER, 'Modern Analysis,' p. 228.

is equal to

$$\frac{1}{t-x}+(-1)^{n+1}\frac{\gamma(\gamma+1)\ldots(\gamma+n)}{(p+n+1)(p+n+2)\ldots(p+2n+1)}\frac{1}{t-x}\left\{\frac{1}{t^{n+1}}\mathrm{F}(-n-1,\ p+n+1,\ \gamma,\ x)\right.$$
$$\mathrm{F}\left(n+1,\ \gamma+n,\ p+2n+1,\ \frac{1}{t}\right)$$
$$+\frac{(n+1)(p-\gamma+n+1)}{(p+2n+1)(p+2n+2)}\frac{1}{t^{n+2}}\mathrm{F}(-n,\ p+n,\ \gamma,\ x)$$
$$\left.\mathrm{F}\left(n+2,\ \gamma+n+1,\ p+2n+3,\ \frac{1}{t}\right)\right\} \quad \ldots \ldots \quad (27).$$

Since both sides of the equation represent analytic functions for all values of γ and p, unless γ or $p+1$ is a negative integer, the restriction that $\mathrm{R}(\gamma)$ and $\mathrm{R}(p+1-\gamma)$ are positive may now be removed.

§ 14. From Theorem I., we find that when n is very large,

$$\mathrm{F}(-n,\ p+n,\ \gamma,\ x)$$

is approximately equal to

$$(-1)^n.\frac{(p+n)(p+n+1)\ldots(p+2n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}x^{\frac{1}{4}(1-2\gamma)}(x-1)^{\frac{1}{4}(2\gamma-2p-1)}\left\{\frac{2x-1}{4}+\tfrac{1}{2}\sqrt{x^2-x}\right\}^{n+\frac{1}{2}p}. \quad (28)$$

for all finite values of x; and

$$\frac{1}{t^{n+1}}\mathrm{F}\left(n+1,\ \gamma+n,\ p+2n+1,\ \frac{1}{t}\right)$$

is approximately equal to

$$t^{\frac{1}{4}(2\gamma-3)}.(t-1)^{\frac{1}{4}(2p-2\gamma-1)}.\{\tfrac{1}{4}(2t-1)+\tfrac{1}{2}\sqrt{t^2-t}\}^{-n-\frac{1}{2}p} \quad \ldots \ldots \quad (29)$$

for all finite values of t.

The second term on the right side of (27), when n is large, is approximately equal to

$$\frac{1}{t-x}t^{-\frac{1}{2}}\left(\frac{x}{t}\right)^{\frac{1}{4}(1-2\gamma)}\left(\frac{x-1}{t-1}\right)^{\frac{1}{4}(2\gamma-2p-1)}\left\{\tfrac{1}{4}(2x-1)+\tfrac{1}{2}\sqrt{x^2-x}\right.$$
$$\left.-\frac{(n+1)(n+p)(n+\gamma)(n+p-\gamma+1)}{\left(n+\frac{p}{2}\right)\left(n+\frac{p+1}{2}\right)^2\left(n+\frac{p+2}{2}\right)}[\tfrac{1}{4}(2t-1)-\tfrac{1}{2}\sqrt{t^2-t}]\right\}$$
$$\times\left[\frac{2x-1+2\sqrt{x^2-x}}{2t-1+2\sqrt{t^2-t}}\right]^{n+\frac{1}{2}p},$$

and vanishes or becomes infinite when n increases indefinitely, according as

$$|2x-1+2\sqrt{x^2-x}|\lesseqgtr|2t-1+2\sqrt{t^2-t}|,$$

that is, according as x lies within or without an ellipse passing through t, which has the points 0 and 1 for foci.

We have the following result.

Theorem VI.—If the point x is in the interior of the ellipse which passes through t and has the points zero and unity for foci,

$$(t-x)^{-1}$$

can be expanded in the series of hypergeometric polynomials

$$\frac{1}{t}\mathrm{F}(0, p, \gamma, x)\mathrm{F}\left(1, \gamma, p+1, \frac{1}{t}\right)$$

$$-\frac{\gamma}{p+1}\frac{1}{t^2}\mathrm{F}(-1, p+1, \gamma, x)\mathrm{F}\left(2, \gamma+1, p+3, \frac{1}{t}\right)+\ldots$$

$$+(-1)^n\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{(p+n)(p+n+1)\ldots(p+2n-1)}\frac{1}{t^{n+1}}\mathrm{F}(-n, p+n, \gamma, x)$$

$$\mathrm{F}\left(n+1, \gamma+n, p+2n+1, \frac{1}{t}\right)$$

$$+\ldots ad\ inf. \qquad (30).$$

If x is outside the ellipse, the series is divergent.

If x is on the ellipse, and $t-x$ is not zero, the sum of n terms of the series, when n is large, oscillates in general between two finite limits. Hence when x is on the ellipse the expansion fails.

§ 15. Multiply both sides of the last equation by $\phi(t)dt$ and integrate round a simple contour enclosing the points $t=0$ and $t=1$ but no singularity of $\phi(t)$.

Theorem VII.—Let $\phi(z)$ be any analytic function which is regular at all points in the interior of an ellipse C, whose foci are at the points

$$z=0 \quad \text{and} \quad z=1.$$

The ellipse is so large that its circumference passes through one (or more) of the singularities of $\phi(z)$. The curve is thus completely defined when $\phi(z)$ is given.

Let p and γ be any quantities, real or complex, subject only to the conditions that the real parts of γ and $p-\gamma+1$ are positive.

Then $\phi(z)$ can be expanded in the infinite series of polynomials

$$a_0\mathrm{F}(0, p, \gamma, z)+a_1\mathrm{F}(-1, p+1, \gamma, z)$$
$$+a_2\mathrm{F}(-2, p+2, \gamma, z)+\ldots+a_n\mathrm{F}(-n, p+n, \gamma, z)+\ldots \qquad (31),$$

where

$$a_n = \frac{(p+2n)\,\Pi(\gamma+n-1)\,\Pi(p+n-1)}{\{\Pi(\gamma-1)\}^2\,\Pi(n)\,\Pi(p-\gamma+n)}\int_0^1 t^{\gamma-1}(1-t)^{p-\gamma}\,\mathrm{F}(-n, p+n, \gamma, t)\,\phi(t)\,dt \quad (32).$$

The series is convergent if z is inside C and divergent if z is outside C. If z is on C, the series is in general oscillatory and the expansion fails.

Moreover, the expansion holds for unrestricted values of p and γ, save that neither $p+1$ nor γ may be a negative integer, if we write

$$a_n = (-1)^n \frac{\gamma(\gamma+1)\dots(\gamma+n-1)}{(p+n)(p+n+1)\dots(p+2n-1)} \frac{1}{2\pi i}$$

$$\int_{C'} \frac{1}{t^{n+1}} F\left(n+1, \gamma+n, p+2n+1, \frac{1}{t}\right)\phi(t)dt. \quad . \quad . \quad . \quad (33),$$

a form equivalent to (32) when the real parts of γ and $p-\gamma+1$ are positive; C′ being a simple closed contour containing the points $t=0$ and $t=1$, but no singularity of $\phi(t)$.

§ 16. The following is a generalisation of (21).

The integral

$$\int^{(0+,1+,0-,1-)} \frac{z^{\gamma-1}(1-z)^{p-\gamma}}{z-t} F(-n, p+n, \gamma, z)\, dz$$

is equal to

$$F(n+1, -p-n+1, 2-\gamma, t)\int^{(0+,1+,0-,1-)} z^{\gamma-2}(1-z)^{p-\gamma} F(-n, p+n, \gamma, z)\, dz \;. \quad (34),$$

the equality holding for unrestricted values of n, p, γ, and t, save that neither $p+1$ nor γ may be a negative integer.

We deduce

Theorem VIII.—Let $\phi(z)$ be any function of z which is regular at all points in the interior of an ellipse C whose foci are at the points $z=0$ and $z=1$. The ellipse passes through one (or more) of the singularities of $\phi(z)$. The curve is thus completely defined when $\phi(z)$ is given.

Further, let p, q, and γ be any constant quantities whatever, real or complex, save that neither $p+1$ nor γ is a negative integer. Then $\phi(z)$ can be expanded in the infinite series of hypergeometric functions

$$a_0 F(p, q, \gamma, z) + a_1 F(p+1, q-1, \gamma, z) + \dots + a_n F(p+n, q-n, \gamma, z) + \dots \quad . \quad (35),$$

where

$$a_n = \frac{(p-q+2n)\,\Pi(\gamma-q+n-1)\,\Pi(p+n-1)}{\{\Pi(\gamma-1)\}^2\,\Pi(n-q)\,\Pi(p-\gamma+n)} \cdot \int^{(0+,1+,0-,1-)} t^{\gamma-1}(1-t)^{p+q-\gamma}$$

$$F(p+n, q-n, \gamma, t)\,\phi(t)\,dt.$$

The series is convergent if z is inside C, and divergent if z is outside C; if z is on C, the series is in general oscillatory and the expansion fails.

§ 17. Expansions in Legendre functions can be deduced from expansions in hypergeometric functions by an appropriate transformation. On account of the special interest of Legendre functions, we give a list of the formulæ obtained,

*When $|z| < 1$,

$$F(p+n,\ 1-p-n,\ 1-m,\ z) = \Pi(-m) \cdot \left(\frac{z}{z-1}\right)^{\frac{1}{2}m} \cdot P^m{}_{p+n-1}(1-2z),$$

and the equality still holds when $|z| > 1$, if $F(p+n,\ 1-p-n,\ 1-m,\ z)$ be replaced by the function derived from it by analytic continuation.

Theorem IX.—Let $\phi(z)$ be any function of z which is regular at all points in the interior of an ellipse C, whose foci are at the points $z = 1$ and $z = -1$.

The ellipse is so large that its circumference passes through one (or more) of the singularities of $\phi(z)$. The curve is thus completely defined when $\phi(z)$ is given.

Let m and p be any constant quantities whatever, real or complex.

Then $\phi(z)$ can be expanded in the infinite series of LEGENDRE's associated functions

$$e^{m\pi i}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}m}\left[a_0 P^m{}_p(z) + a_1 P^m{}_{p+1}(z) + \ldots + a_n P^m{}_{p+n}(z) + \ldots\right] \quad . \quad . \quad . \quad (36),$$

where

$$a_n = -\tfrac{1}{2}(2p+2n+1)\frac{\Pi(p-m+n)}{\Pi(p+m+n)}\int^{(1+,\ -1+,\ 1-,\ -1-)}\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}m} P^m{}_{p+n}(t)\,\phi(t)\,.\,dt.$$

The series is convergent if z is inside C and divergent if z is outside C. If z is on C, the series is, in general, oscillatory and the expansion fails.

Theorem X.—If the point μ is in the interior of the ellipse which passes through ρ, and has the points $\mu = 1$, $\mu = -1$, for foci, and if m is any constant quantity whatever, real or complex,

$$(\rho-\mu)^{-1}$$

can be expanded in the infinite series of LEGENDRE's associated functions,

$$e^{m\pi i}\left(\frac{\mu-1}{\mu+1} \div \frac{\rho-1}{\rho+1}\right)^{\frac{1}{2}m}\{P_0{}^m(\mu)Q_0{}^{-m}(\rho) + 3P_1{}^m(\mu)Q_1{}^{-m}(\rho)$$
$$+ \ldots + (2n+1)P_n{}^m(\mu)Q_n{}^{-m}(\rho) + \ldots\} \quad . \quad . \quad . \quad . \quad . \quad (37).$$

If μ is outside the ellipse, the series is divergent. If μ is on the ellipse, the series is, in general, oscillatory and the expansion fails.

Theorem XI.—Let $\phi(z)$ be any function which is regular at all points in the interior of an ellipse C, whose foci are at the points $z = 1$ and $z = -1$.

The ellipse is so large that its circumference passes through one (or more) of the singularities of $\phi(z)$. The curve is thus completely defined when $\phi(z)$ is given.

Let m be any constant quantity, such that the real part of m lies between 1 and -1.

* HOBSON, "On a Type of Spherical Harmonics of unrestricted Degree, Order and Argument," 'Phil. Trans.,' 1896, Series A, vol. 187, p. 451 (5). The notation used by HOBSON, including the definitions of the functions $P_n{}^m(\mu)$, $Q_n{}^m(\mu)$, will be adopted.

Then $\phi(z)$ can be expanded* in the infinite series of LEGENDRE'S associated functions

$$e^{m\pi i}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}m}\{a_0 P_0^m(z)+a_1 P_1^m(z)+\ldots+a_n P_n^m(z)+\ldots\} \quad . \quad . \quad . \quad . \quad (38),$$

where

$$a_n = \tfrac{1}{2}(2n+1)\frac{\Pi(n-m)}{\Pi(n+m)}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{\frac{1}{2}m} P_n^m(t)\phi(t)\,dt.$$

The series is convergent if z is inside C, divergent if z is outside C, and, in general, oscillatory if z is on C, in which case the expansion fails.

* The position of this result in the historical development of the subject is noteworthy. If m is a real positive integer as well as n, $P_n^m(z)$ vanishes so long as $n<m$.

With the further restriction on $\phi(z)$ that $(1+z)^{1-\frac{1}{2}m}\phi(z)$ vanishes when $z=-1$, we find that

$$\phi(z) = e^{m\pi i}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}m}\sum_{n=m}^{\infty} a_n P_n^m(z),$$

where

$$a_n = \tfrac{1}{2}(2n+1)\frac{\Pi(n-m)}{\Pi(n+m)}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{\frac{1}{2}m} P_n^m(t)\,\phi(t)\,dt.$$

This result is given by HEINE ('Kugelfunctionen,' 2nd edition, p. 252).

In his notation

$$f(z) \text{ denotes } \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}m}\phi(z).$$

If in the last equation $m=0$, we have the well-known expansion, valid within the ellipse C, in terms of the simple Legendre functions,

$$\phi(z) = a_0 P_0(z) + a_1 P_1(z) + \ldots + a_n P_n(z) + \ldots,$$

where

$$a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{1} P_n(t)\,\phi(t)\,dt$$

(WHITTAKER, 'Modern Analysis,' p. 230).